Technical Notes

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Extended Integral Equation Method for Transonic Flows

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Introduction

IN the standard integral equation method[1,2] for calculating transonic flows, the field (double) integral that is a feature of such methods is reduced to a line integral over the aerofoil chord by representing the transverse variation of the velocity in terms of the velocity on the aerofoil surface using a fairly arbitrary approximation function. The resulting integral equation for the surface velocity can be solved without much difficulty. This approximate evaluation of the field integral, although adequate for subcritical flows, is not sufficiently accurate to give satisfactory results for supercritical flows when shock waves are present. In the extended integral equation method presented in the section, an alternative means of evaluating the field integral is developed, in which the fundamental integral equation is used to find velocities in the flowfield in addition to the surface velocities. This results in a considerable improvement in the accuracy of the integral equation method.

Analysis

Integral equation methods[1] for calculating transonic flows were developed before efficient digital computers were available and consequently involve a minimum of numerical work. In the integral equation method the nonlinear transonic small disturbance equation is written in integral form consisting of a line integral over the aerofoil chord and a double integral, involving the second-order terms, over the entire flowfield. The inaccuracies introduced into the fundamental integral equation through the approximate evaluation of the field integral are responsible for the unsatisfactory comparison between the results of the integral equation method[1,2] and the results of the finite difference methods.[3,4] The main advantage of integral equation methods over finite difference methods is that, in principle, the numerical solution of the integral equation is easier than the solution of the differential equation, typified by rapid convergence. The main disadvantage of existing solutions[1,2] of the transonic integral equation is the unsatisfactory results due to the rather primitive, and inaccurate, approximation of the field integral. If the accuracy of the evaluation of the field integral can be improved without sacrificing too much numerical simplicity, it is suggested that integral equation methods will provide a useful research tool in many problems of transonic aerodynamics. Only the flow over a symmetric aerofoil at zero incidence is considered here, but in general the extended integral equation method can be applied to a wide range of problems.

For a freestream Mach number M_∞ and a transonic parameter k the fundamental integral equation[1] for symmetric aerofoils at zero incidence is

$$\bar{u}(\bar{x},\bar{z}) - \bar{u}^2(\bar{x},\bar{z})/2 = \bar{u}_{TL}(\bar{x},\bar{z}) + I_T(\bar{x},\bar{z},\bar{x}_s) \tag{1}$$

where $\bar{u}(\bar{x},\bar{z})$ is related to the streamwise perturbation velocity $u(x,z)$ by

$$\bar{u}(\bar{x},\bar{z}) = [(\gamma+1)k/(1-M_\infty^2)]\, u(x,z) \tag{2}$$

and

$$\bar{u}_{TL}(\bar{x},\bar{z}) = \frac{1}{\pi}\int_0^1 \frac{\bar{Z}_T'(\xi)\,(\bar{x}-\xi)}{(\bar{x}-\xi)^2+\bar{z}^2}\,d\xi \tag{3}$$

where $\bar{Z}_T(\bar{x}) = [(\gamma+1)k/(1-M_\infty^2)^{3/2}]z_T(x)$; $z = z_T(x)$ denotes the thickness distribution of the aerofoil. $I_T(\bar{x},\bar{z},\bar{x}_s)$ is a field integral defined by

$$\begin{aligned} I_T(\bar{x},\bar{z},\bar{x}_s) = \\ -\frac{1}{4\pi}\lim_{\substack{\epsilon\to 0\\ \delta\to 0}} \Big\{ &\int_{-\infty}^{\bar{x}-\epsilon}\Big(\int_0^\infty \psi_{\bar{\xi}_{\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\bar{u}^2(\bar{\xi},\bar{\zeta})d\bar{\zeta}\Big)d\bar{\xi} \\ &+\int_{\bar{x}+\epsilon}^{\bar{x}_s-\delta}\Big(\int_0^\infty \psi_{\bar{\xi}_{\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\bar{u}^2(\bar{\xi},\bar{\zeta})d\bar{\zeta}\Big)d\bar{\xi} \\ &+\int_{\bar{x}_s+\delta}^{\infty}\Big(\int_0^\infty \psi_{\bar{\xi}_{\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\bar{u}^2(\bar{\xi},\bar{\zeta})d\bar{\zeta}\Big)d\bar{\xi} \\ &+\int_{-\infty}^{\bar{x}_s-\delta}\Big(\int_{-\infty}^0 \psi_{\bar{\xi}_{\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\bar{u}^2(\bar{\xi},\bar{\zeta})d\bar{\zeta}\Big)d\bar{\xi} \\ &+\int_{\bar{x}_s+\delta}^{\infty}\Big(\int_{-\infty}^0 \psi_{\bar{\xi}_{\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\bar{u}^2(\bar{\xi},\bar{\zeta})d\bar{\zeta}\Big)d\bar{\xi}\Big\} \end{aligned} \tag{4}$$

where $\bar{x}_s$ is the shock location and

$$\psi(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) = \log_e\{(\bar{x}-\bar{\xi})^2+(\bar{z}-\bar{\zeta})^2\}^{1/2}$$

The second-order equation, Eq. (1), has two "solutions" given by

$$\bar{u}(\bar{x},\bar{z}) = 1 \pm \{1-2[\bar{u}_{TL}(\bar{x},\bar{z}) + I_T(\bar{x},\bar{z},\bar{x}_s)]\}^{1/2}$$

Since sonic conditions exist when $(\bar{u}(\bar{x},\bar{z}) = 1)$ the upper root denotes a supersonic solution and the lower root denotes a subsonic solution. A shock is represented by a discontinuous jump from the upper root to the lower root. If a shock wave is present in the flow, the shock location $\bar{x}_s$ is found by ensuring a finite acceleration at the sonic line $\bar{x}_0(\bar{z})$. This leads to the following set of equations:

$$\{\bar{u}_{TL}(\bar{x},\bar{z}) + I_T(\bar{x},\bar{z},\bar{x}_s)\}\,|_{\bar{x}=\bar{x}_0(\bar{z})} = 1/2 \tag{5a}$$

$$\frac{\partial}{\partial\bar{x}}\{\bar{u}_{TL}(\bar{x},\bar{z}) + I_T(\bar{x},\bar{z},\bar{x}_s)\}\,|_{\bar{x}=\bar{x}_0(\bar{z})} = 0 \tag{5b}$$

$$\frac{\partial}{\partial\bar{z}}\{\bar{u}_{TL}(\bar{x},\bar{z}) + I_T(\bar{x},\bar{z},\bar{x}_s)\}\,|_{\bar{x}=\bar{x}_0(\bar{z})} = 0 \tag{5c}$$

Equations (5) are sufficient to give the sonic line $\bar{x}_0(\bar{z})$, the normal shock location $\bar{x}_s$, and the slope of the sonic line $d\bar{x}_0(\bar{z})/d\bar{z}$.

Received May 14, 1974; revision received December 2, 1974.
Index category: Subsonic and Transonic Flow.

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In order to solve the fundamental integral equation, Eq. (1), the field integral $I_T(\bar{x},\bar{z},\bar{x}_s)$, defined by Eq. (4), needs to be evaluated and this requires a knowledge of the variation of $\bar{u}(\bar{\xi}, \bar{\zeta})$ over the entire flowfield. An approximate, although sufficiently accurate, evaluation of $I_T(\bar{x},\bar{z},\bar{x}_s)$ is possible however, if $\bar{u}(\bar{\xi},\bar{\zeta})$ is known only at specified points in the flowfield and some kind of interpolation function is used to express $\bar{u}(\bar{\xi},\bar{\zeta})$ in the rest of the flowfield. It should be noted that the practical requirement is an accurate evaluation of $\bar{u}(\bar{\xi},\bar{\zeta})$ on the aerofoil surface, and hence an acceptable solution to Eqs. (1) and (5) need not necessarily be very accurate away from the surface, provided the surface velocities are calculated to the desired accuracy.

Let the flowfield be divided into $2N$ strips ($\bar{z}$ = constant) as shown in Fig. 1; since the present analysis is restricted to symmetric flows only the upper half plane need be considered. The jth strip is defined by

$$\bar{\zeta}_j \leq \bar{\zeta} \leq \bar{\zeta}_{j+1} \quad (j=1, N) \tag{6}$$

and $\bar{\zeta}_1 = 0$. The Nth strip in each half plane is assumed to stretch from $\bar{\zeta} = \bar{\zeta}_N$ to $\bar{\zeta} = \infty$; that is $\bar{\zeta}_{N+1}\infty = 1$. It is assumed that the velocity $\bar{u}(\bar{\xi},\bar{\zeta})$ is known on each strip edge $\bar{\zeta} = \bar{\zeta}_j$ $(j=1, N+1)$ and linear variation of $\bar{u}^2(\bar{\xi}, \bar{\zeta})$ is used in the first $(N$-1) strips; thus

$$\bar{u}^2(\bar{\xi},\bar{\zeta}) = \bar{u}^2(\bar{\xi},\bar{\zeta}_j) + \frac{\{\bar{u}^2(\bar{\xi},\bar{\zeta}_{j+1}) - \bar{u}^2(\bar{\xi},\bar{\zeta}_j)\}}{(\bar{\zeta}_{j+1} - \bar{\zeta}_j)} (\bar{\zeta} - \bar{\zeta}_j) \tag{7a}$$

for $\bar{\zeta}_j \leq \bar{\zeta} \leq \bar{\zeta}_{j+1}$, $j<N$. In the semi-infinite N^{th} strip in each half plane it is assumed that

$$\bar{u}^2(\bar{\xi},\bar{\zeta}) = \frac{\bar{u}^2(\bar{\xi},\bar{\zeta}_N)}{\{1 + (\bar{\zeta} - \bar{\zeta}_N)/b(\bar{\xi},\bar{\zeta}_N)\}^2}, \quad \bar{\zeta} > \bar{\zeta}_N \tag{7b}$$

where

$$b(\bar{\xi},\bar{\zeta}_N) = \frac{-2(\bar{\zeta}_N - \bar{\zeta}_{N-1})\bar{u}^2(\bar{\xi},\bar{\zeta}_N)}{\{\bar{u}^2(\bar{\xi},\bar{\zeta}_N) - \bar{u}^2(\bar{\xi},\bar{\zeta}_{N-1})\}} \tag{8}$$

It is also assumed that for the purpose of evaluating $I_T(\bar{x},\bar{z}, \bar{x}_s)$, the velocity $\bar{u}(\bar{\xi},\bar{\zeta})$ is zero for $(\bar{\xi}<0,\ \bar{\xi}>1)$. Using Eq. (7), the integration with respect to $\bar{\zeta}$ in the field integral $I_T(\bar{x},\bar{z},\bar{x}_s)$ can be performed; the accuracy of this integration can be increased simply by increasing the number of strips. The evaluation of the subsequent line integral in the $\bar{\xi}$ direction can be obtained by standard methods[1,2] in terms of the velocity $\bar{u}(\bar{x}_i, \bar{\zeta}_j)$, $(i=1, m)$ where $\bar{x}_i$, $(i=1, m)$ are specified values of $\bar{x}$ $(0 \leq \bar{x} \leq 1)$. On putting $(\bar{x}=\bar{x}_i)$, $(\bar{z}=\bar{z}_j=\bar{\zeta}_j)$ the integral equation, Eq. (1) becomes

$$\bar{u}(\bar{x}_i, \bar{z}_j) - \tfrac{1}{2}\bar{u}^2(\bar{x}_i, \bar{z}_j) = \bar{u}_{TL}(\bar{x}_i, \bar{z}_j) + I_T(\bar{x}_i, \bar{z}_j, \bar{x}_s) \tag{9}$$

and $I_T(\bar{x}_i, \bar{z}_j, \bar{x}_s)$ is a function of $\bar{u}(\bar{x}_i,\bar{z}_j)$.

For subcritical flows, Eq. (9) can be solved directly using an iterative technique. For supercritical flows, Eq. (9) must be solved subject to the conditions Eq. (5).

Results

In order to compare results, the similarity parameter k used by Murman and Cole[3] is used; thus $K = (1 - M_\infty^2)/(\tau k)^{2/3}$

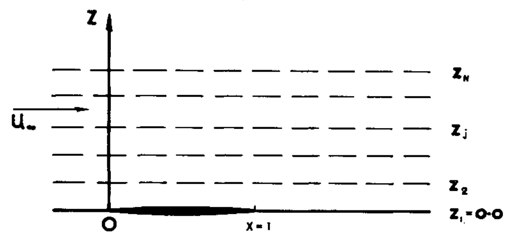

Fig. 1 Arrangement of strips for a half-plane.

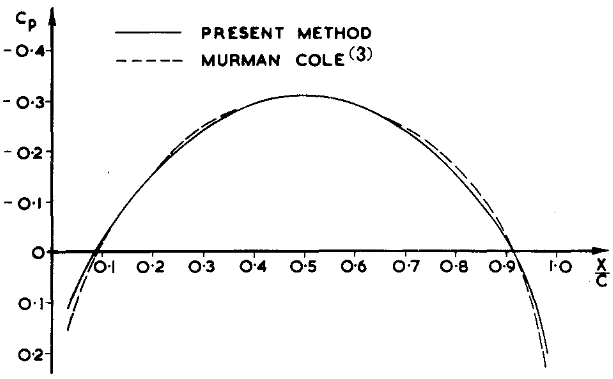

Fig. 2 Pressure distribution around a 6% biconvex aerofoil, $M_\infty = 0.8$ ($K = 2.94$).

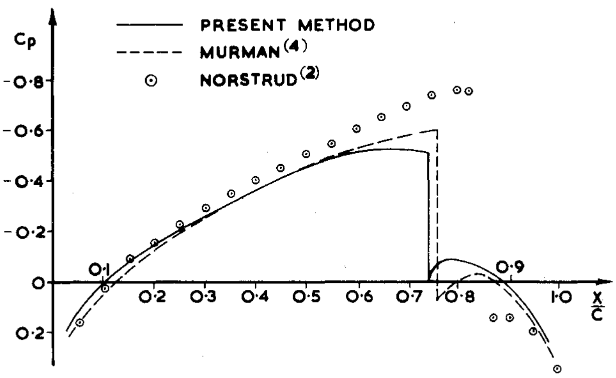

Fig. 3 Pressure distribution around a 6% biconvex aerofoil, $M_\infty = 0.87$ ($K = 1.8$).

where τ is the thickness-chord ratio of the aerofoil. The pressure distributions around biconvex aerofoils in both subcritical flow ($K=2.94$) and in supercritical flow when shock waves are present ($K=1.8$) are calculated. The subcritical result ($K=2.94$) is shown in Fig. 2 and is compared to the finite difference result of Murman and Cole.[3] There is good agreement between these two results.

The pressure distribution for the supercritical case ($K = 1.8$) is shown in Fig. 3 and is compared to the finite difference results of Murman[4] and to the "standard" integral equation results of Norstrud.[2] It can be seen that both the pressure distribution and the predicted shock location of the present results agree satisfactorily with the results of Murman[4] and are a considerable improvement on the "standard" integral equation results.

The computing time of the extended integral equation method is slightly greater than that required by the finite difference methods. The bulk of this computing time is used in the evaluation of the field integral in the relatively unimportant semi-infinite outer strip; this is due to the nonlinear dependence of the approximation function, Eq. (7b) on $\bar{u}^2(\bar{\xi}, \bar{\zeta})$ through the gradient factor $b(\bar{\xi},\bar{\zeta}_N)$ which prohibits storage of the resulting influence function. If a satisfactory alternative function to that of Eq. (7b) can be devised that is linear in $b(\bar{\xi},\bar{\zeta}_N)$, then the computing time will be reduced considerably. The extended integral equation method would probably then be computationally superior to the finite difference methods.

References

[1] Spreiter, J. R. and Alksne, A. Y., "Theoretical Predictions of Pressure Distributions on Non-Lifting Airfoils at High Subsonic Speeds," Rept. 1217, 1955, NACA.

[2] Norstrud, H., "High Speed Flow past Wings," CR-2246, April 1973, NASA.

[3] Murman, E. and Cole, J. D., "Calculation of Plane Steady Transonic Flows," *AIAA Journal*, Vol. 9, June 1971, pp. 114-121.

[4] Murman, E., "Analysis of Embedded Shock Waves Calculated by Relaxation Methods," *AIAA Journal*, Vol. 13, May 1974, pp. 626-633.